CBSE 2025 EXAMINATIONS
XII MATHEMATICS (041)
Series ZXW5Y ♦ Q.P. Code 65/5/1; 65/5/2; 65/5/3

Time Allowed : 180 Minutes **Max. Marks : 80**

General Instructions :

1. This Question paper contains **five sections - A, B, C, D and E**. Each section is compulsory. However, there are **internal choices** in some questions.
2. Section A has **18 MCQs** and **02 Assertion-Reason (A-R)** based questions of **1 mark** each.
   Section B has **05 questions** of **2 marks** each.
   Section C has **06 questions** of **3 marks** each.
   Section D has **04 questions** of **5 marks** each.
   Section E has **03 Case-study / Source-based / Passage-based** questions with **sub-parts** (**4 marks** each).
3. There is no overall choice. However, **internal choice** has been provided in
   - **02 Questions of Section B**
   - **03 Questions of Section C**
   - **02 Questions of Section D**
   - **02 Questions of Section E**

   You have to attempt only one of the alternatives in all such questions.

☑ Q.P. Code - 65/5/1

SECTION A

This section has 20 multiple choice questions of 1 mark each.

01. If $A=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$, then A^3 is

(a) $3\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$ (b) $\begin{bmatrix}125&0&0\\0&125&0\\0&0&125\end{bmatrix}$ (c) $\begin{bmatrix}15&0&0\\0&15&0\\0&0&15\end{bmatrix}$ (d) $\begin{bmatrix}5^3&0&0\\0&5&0\\0&0&5\end{bmatrix}$

Sol. (b) Note that $A=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}=5\,I$.

So, $A^3=AAA=(5\,I)(5\,I)(5\,I)=125\,I^3=125\,I=\begin{bmatrix}125&0&0\\0&125&0\\0&0&125\end{bmatrix}$.

02. If $P(A\cup B)=0.9$ and $P(A\cap B)=0.4$, then $P(\bar{A})+P(\bar{B})$ is

(a) 0.3 (b) 1 (c) 1.3 (d) 0.7

Sol. (d) $P(\bar{A})+P(\bar{B})=[1-P(A)]+[1-P(B)]=2-[P(A)+P(B)]=2-[P(A\cup B)+P(A\cap B)]$

$=2-(0.9+0.4)=0.7$.

03. If $A=\begin{bmatrix}1&2&3\\-4&3&7\end{bmatrix}$ and $B=\begin{bmatrix}4&3\\-1&2\\0&5\end{bmatrix}$, then the correct statement is

(a) Only AB is defined (b) Only BA is defined

(c) AB and BA, both are defined (d) AB and BA, both are not defined

Sol. (c) AB and BA, both are defined.

04. If $\begin{vmatrix} 2x & 5 \\ 12 & x \end{vmatrix} = \begin{vmatrix} 6 & -5 \\ 4 & 3 \end{vmatrix}$, then the value of x is

(a) 3 (b) 7 (c) ± 7 (d) ± 3

Sol. (c) Expanding the det. on both the sides, we get $2x^2 - 60 = 18 - (-20)$

$\Rightarrow 2x^2 = 38 + 60 = 98$

$\Rightarrow x^2 = 49$

$\therefore x = \pm 7$.

05. If $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of a is

(a) 1 (b) –1 (c) ± 1 (d) 0

Sol. (c) As f is continuous at $x = 0$, so $f(0) = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$.

Now $1 = \lim\limits_{x \to 0} \dfrac{\sin^2 ax}{x^2}$

$\Rightarrow 1 = \lim\limits_{ax \to 0} \dfrac{\sin^2 ax}{(ax)^2} \times a^2$

$\Rightarrow 1 = 1^2 \times a^2$

$\therefore a = \pm 1$.

06. If $A = \left[a_{ij}\right]$ is a 3×3 diagonal matrix such that $a_{11} = 1$, $a_{22} = 5$ and $a_{33} = -2$, then $|A|$ is

(a) 0 (b) –10 (c) 10 (d) 1

Sol. (b) The det. value of a diagonal matrix can be obtained by multiplying all its diagonal elements.

Hence, $|A| = a_{11} \times a_{22} \times a_{33} = 1 \times 5 \times (-2) = -10$.

07. The principal value of $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$ is

(a) $-\dfrac{\pi}{3}$ (b) $-\dfrac{2\pi}{3}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{2\pi}{3}$

Sol. (d) $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right) = \pi - \cot^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$.

08. If $\begin{bmatrix} 4+x & x-1 \\ -2 & 3 \end{bmatrix}$ is a singular matrix, then the value of x is

(a) 0 (b) 1 (c) –2 (d) –4

Sol. (c) Since the given matrix is a singular matrix so, $\begin{vmatrix} 4+x & x-1 \\ -2 & 3 \end{vmatrix} = 0$

$\Rightarrow (4+x)(3) - (x-1)(-2) = 0$

$\Rightarrow 12 + 3x + 2x - 2 = 0$

$\Rightarrow 5x + 10 = 0 \qquad \therefore x = -2$.

09. If $f(x) = \{[x], x \in \mathbb{R}\}$ is the greatest integer function, then the correct statement is

(a) f is continuous but not differentiable at $x = 2$

(b) f is neither continuous nor differentiable at $x = 2$

(c) f is continuous as well as differentiable at $x = 2$

(d) f is not continuous but differentiable at $x = 2$

Sol. (b) f is neither continuous nor differentiable at $x = 2$.

10. The slope of the curve $y = -x^3 + 3x^2 + 8x - 20$ is maximum at

(a) (1, –10) (b) (1, 10) (c) (10, 1) (d) (–10, 1)

Sol. (a) Slope of the curve $= \frac{dy}{dx} = -3x^2 + 6x + 8 = m$

Now $\frac{dm}{dx} = -6x + 6$ and $\frac{d^2m}{dx^2} = -6$.

For $\frac{dm}{dx} = -6x + 6 = 0 \Rightarrow x = 1$.

Clearly, m is maximum at $x = 1$, as $\left(\frac{d^2m}{dx^2}\right)_{\text{at } x=1} = -6 < 0$.

When $x = 1$, $y = -1^3 + 3\times1^2 + 8\times1 - 20 = -1 + 3 + 8 - 20 = -10$.

Therefore, the slope of given curve is maximum at (1, –10).

11. $\int \sqrt{1+\sin x}\, dx$ is equal to

(a) $2\left(-\sin\frac{x}{2} + \cos\frac{x}{2}\right) + C$ (b) $2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + C$

(c) $-2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + C$ (d) $2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + C$

Sol. (b) Note that, $\sqrt{1+\sin x} = \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} = \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} = \sin\frac{x}{2} + \cos\frac{x}{2}$.

So, $\int \sqrt{1+\sin x}\, dx = \int\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx = -\frac{\cos\frac{x}{2}}{\frac{1}{2}} + \frac{\sin\frac{x}{2}}{\frac{1}{2}} + C = 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + C$.

12. $\int_0^{\pi/2} \cos x . e^{\sin x} dx$ is equal to

(a) 0 (b) $1 - e$ (c) $e - 1$ (d) e

Sol. (c) Put $e^{\sin x} = t \Rightarrow e^{\sin x}\cos x\, dx = dt$.

So, $\int e^{\sin x}\cos x\, dx = \int dt = t + C = e^{\sin x} + C$.

Hence, $\int_0^{\pi/2} \cos x . e^{\sin x} dx = \left[e^{\sin x} + C\right]_0^{\frac{\pi}{2}}$

$$= \left[e^{\sin\frac{\pi}{2}} + C\right] - \left[e^{\sin 0} + C\right] = \left[e^1 + C\right] - \left[e^0 + C\right]$$

$$= e - 1.$$

Here we can ignore "constant of integration, C" as it is a definite integral.

13. The area of the region enclosed between the curve $y = x|x|$, x-axis, $x = -2$ and $x = 2$ is

(a) $\frac{8}{3}$ (b) $\frac{16}{3}$ (c) 0 (d) 8

Sol. (b) We have $y = x|x| = \begin{cases} x^2, \text{ if } x \geq 0 \\ -x^2, \text{ if } x < 0 \end{cases}$, $x = -2$ and $x = 2$.

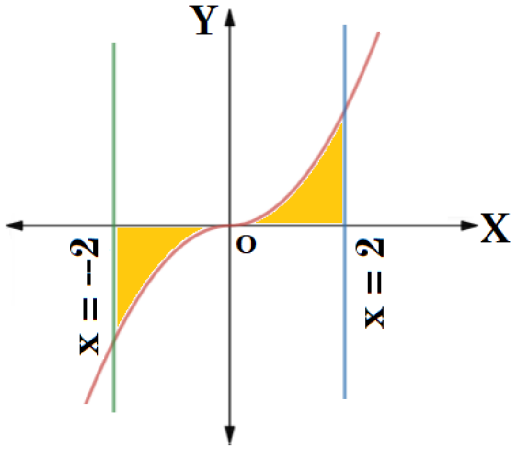
$$\text{Required area} = \left|\int_{-2}^{0} -x^2\,dx\right| + \int_{0}^{2} x^2\,dx$$

$$\Rightarrow \quad = \left|-\frac{1}{3}\left[x^3\right]_{-2}^{0}\right| + \frac{1}{3}\left[x^3\right]_{0}^{2}$$

$$\Rightarrow \quad = \left|-\frac{1}{3}[0-(-8)]\right| + \frac{1}{3}[8-0]$$

$$\Rightarrow \quad = \left|-\frac{8}{3}\right| + \frac{8}{3}$$

$$\Rightarrow \quad = \frac{16}{3}\ \text{Sq. units}.$$



14. The integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ is

(a) $e^{-\frac{1}{\sqrt{x}}}$ (b) $e^{\frac{2}{\sqrt{x}}}$ (c) $e^{2\sqrt{x}}$ (d) $e^{-2\sqrt{x}}$

Sol. (c) Rewriting the given D.E., we get $\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

On comparing with $\frac{dy}{dx} + y\,P(x) = Q(x)$, we get $P(x) = \frac{1}{\sqrt{x}}$.

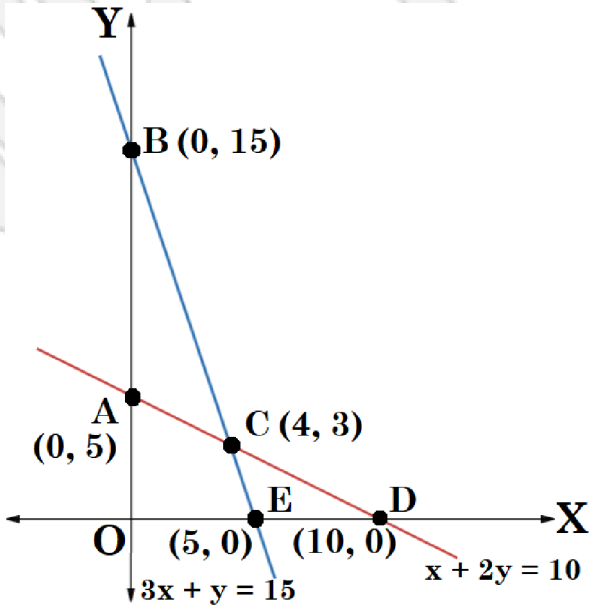
So, integrating factor $(I.F.) = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$.

15. The sum of the order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}$ is

(a) 2 (b) $\frac{5}{2}$ (c) 3 (d) 4

Sol. (c) For the D.E., the order is 2 and degree is 1. So, the required sum will be $2+1=3$.

16. For a Linear Programming Problem (LPP), the given objective function $Z = 3x + 2y$ is subject to constraints $x + 2y \le 10$, $3x + y \le 15$; $x, y \ge 0$.



The correct feasible region is

(a) ABC (b) AOEC
(c) CED (d) Open unbounded region BCD

Sol. (b) The correct feasible region is AOEC.

17. Let $\vec{a}$ be a position vector whose tip is the point (2, –3). If $\overrightarrow{AB} = \vec{a}$, where coordinates of A are (–4, 5), then the coordinates of B are
(a) (–2, –2) (b) (2, –2) (c) (–2, 2) (d) (2, 2)

Sol. (c) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{a}$

$\Rightarrow \overrightarrow{OB} - (-4\hat{i} + 5\hat{j}) = 2\hat{i} - 3\hat{j}$ [$\because \vec{a} = 2\hat{i} - 3\hat{j}$

$\Rightarrow \overrightarrow{OB} = 2\hat{i} - 3\hat{j} + (-4\hat{i} + 5\hat{j}) = -2\hat{i} + 2\hat{j}$.

Therefore, the coordinates of B are (–2, 2).

18. The respective values of $|\vec{a}|$ and $|\vec{b}|$, if given $(\vec{a} - \vec{b}).(\vec{a} + \vec{b}) = 512$ and $|\vec{a}| = 3|\vec{b}|$, are
(a) 48 and 16 (b) 3 and 1 (c) 24 and 8 (d) 6 and 2

Sol. (c) $(\vec{a} - \vec{b}).(\vec{a} + \vec{b}) = 512$ implies, $|\vec{a}|^2 - |\vec{b}|^2 = 512$

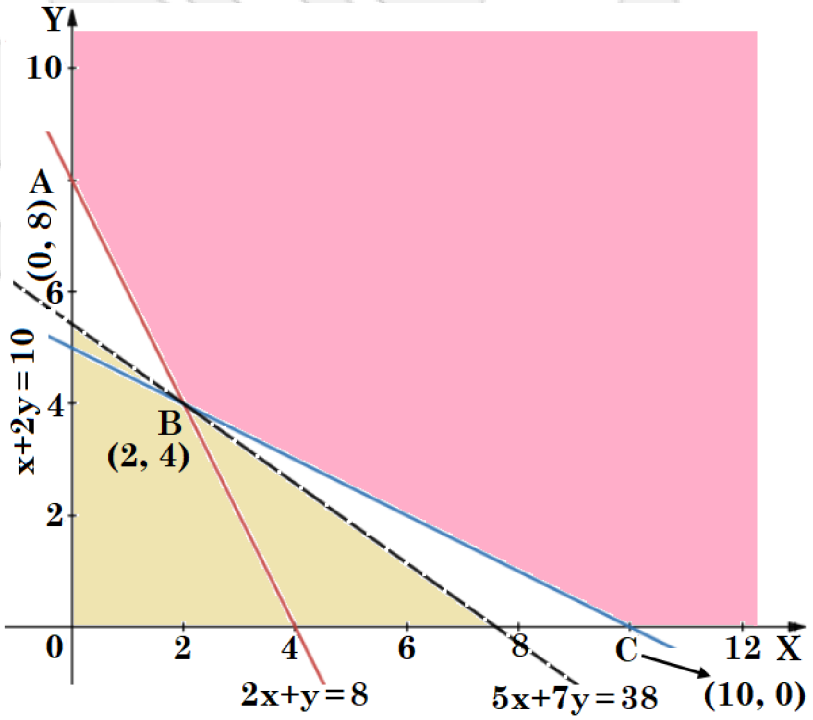
$\Rightarrow \left[3|\vec{b}|\right]^2 - |\vec{b}|^2 = 512$ $\Rightarrow 8 \times |\vec{b}|^2 = 512$ $\Rightarrow |\vec{b}|^2 = 64$

$\therefore |\vec{b}| = 8$, $|\vec{a}| = 3|\vec{b}| = 3 \times 8 = 24$.

Direction : *Question numbers* ***19*** *and* ***20*** *are Assertion (A) and Reason (R) based questions carrying* ***1 mark*** *each. Two statements are given, one labeled Assertion (A) and other labeled Reason (R). Select the correct answer from the options given below.*

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
(b) Both Assertion (A) and Reason (R) are true and Reason (R) is **not** the correct explanation of Assertion (A).
(c) Assertion (A) is true but Reason (R) is false.
(d) Assertion (A) is false but Reason (R) is true.

19. **Assertion (A) :** The shaded portion of the graph represents the feasible region for the given Linear Programming Problem (LPP).



Min $Z = 50x + 70y$

subject to constraints

$2x + y \geq 8,\ x + 2y \geq 10;\ x,\ y \geq 0$.

$Z = 50x + 70y$ has a minimum value $= 380$ at B(2, 4).

Reason (R) : The region representing $50x + 70y < 380$ does not have any point common with the feasible region.

Sol. (a) Here both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

20. **Assertion (A) :** Let $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$. If $f : A \to A$ be defined as $f(x) = x^2$, then f is not an onto function.

Reason (R) : If $y = -1 \in A$, then $x = \pm\sqrt{-1} \notin A$.

Sol. (a) Here both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A). Note that, $y = -1 \in A$ (codomain) has no pre-image in A (domain). So, f is not onto, as Range $\neq$ Codomain.

SECTION B

This section has 5 Very Short Answer questions of 2 marks each.

21. Find the domain of the function $f(x) = \cos^{-1}(x^2 - 4)$.

Sol. For $\cos^{-1}(x^2 - 4)$ to be defined, $x^2 - 4 \in [-1, 1]$ i.e., $-1 \leq x^2 - 4 \leq 1$

$\Rightarrow 3 \leq x^2 \leq 5$

$\therefore -\sqrt{5} \leq x \leq -\sqrt{3}$ or $\sqrt{3} \leq x \leq \sqrt{5}$

Therefore, domain of $f(x) = x \in [-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$.

22. Surface area of a balloon (spherical), when air is blown into it, increases at a rate of 5 mm^2/s. When the radius of the balloon is 8 mm, find the rate at which the volume of the balloon is increasing.

Sol. Surface area of spherical balloon, $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \times \frac{dr}{dt}$

$\Rightarrow 5 = 8\pi r \times \frac{dr}{dt} \qquad \Rightarrow \frac{dr}{dt} = \frac{5}{8\pi r}$ ...(i)

Now volume, $V = \frac{4}{3}\pi r^3 \qquad \Rightarrow \frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt} = 4\pi r^2 \times \frac{5}{8\pi r} = \frac{5r}{2}$ [By (i)

When $r = 8$ mm, $\frac{dV}{dt} = \frac{5 \times 8}{2} = 20\ mm^3/s$.

23. (a) Differentiate $\frac{\sin x}{\sqrt{\cos x}}$ with respect to x.

OR

(b) If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

Sol. (a) Let $y = \frac{\sin x}{\sqrt{\cos x}}$

Differentiating w.r.t. x, $\frac{dy}{dx} = \frac{\sqrt{\cos x}\,\frac{d}{dx}(\sin x) - \sin x\,\frac{d}{dx}\left[\sqrt{\cos x}\right]}{\left[\sqrt{\cos x}\right]^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\cos x} \times \cos x - \sin x \times \frac{1}{2\sqrt{\cos x}} \times (-\sin x)}{\cos x} = \frac{2\cos^2 x + \sin^2 x}{2\cos x\sqrt{\cos x}}$$

$$\therefore \frac{dy}{dx} = \frac{\cos^2 x + 1}{2\cos x\sqrt{\cos x}} \qquad [\because \sin^2 x + \cos^2 x = 1$$

OR

(b) $y = 5\cos x - 3\sin x$ …(i)

On differentiating w.r.t. x, $\frac{dy}{dx} = -5\sin x - 3\cos x$

$$\Rightarrow \frac{d^2y}{dx^2} = -5\cos x + 3\sin x = -y \qquad [\text{By (i)}$$

$$\therefore \frac{d^2y}{dx^2} + y = 0.$$

24. (a) Find a vector of magnitude 5 which is perpendicular to both the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $4\hat{i} + 3\hat{j} - 2\hat{k}$.

OR

(b) Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}, \vec{a} \neq \vec{0}$. Show that $\vec{b} = \vec{c}$.

Sol. (a) $(3\hat{i} - 2\hat{j} + \hat{k}) \times (4\hat{i} + 3\hat{j} - 2\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 4 & 3 & -2 \end{vmatrix} = \hat{i} + 10\hat{j} + 17\hat{k}$

$$\text{Required vector} = 5\left(\pm \frac{(3\hat{i} - 2\hat{j} + \hat{k}) \times (4\hat{i} + 3\hat{j} - 2\hat{k})}{\left|(3\hat{i} - 2\hat{j} + \hat{k}) \times (4\hat{i} + 3\hat{j} - 2\hat{k})\right|} \right) = \pm 5\left(\frac{\hat{i} + 10\hat{j} + 17\hat{k}}{\sqrt{1 + 100 + 289}} \right)$$

$$= \pm 5\left(\frac{\hat{i} + 10\hat{j} + 17\hat{k}}{\sqrt{390}} \right) = \pm\left(\frac{5\hat{i} + 50\hat{j} + 85\hat{k}}{\sqrt{390}} \right).$$

OR

(b) $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ implies, $\vec{a}.\vec{b} - \vec{a}.\vec{c} = 0$

$\Rightarrow \vec{a}.(\vec{b} - \vec{c}) = 0$

$\Rightarrow$ Either $\vec{a} \perp (\vec{b} - \vec{c})$ or $\vec{b} - \vec{c} = \vec{0}$ …(i) $[\because \vec{a} \neq \vec{0}$

Also, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ implies, $\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$

$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$

$\Rightarrow$ Either $\vec{a} \parallel (\vec{b} - \vec{c})$ or $\vec{b} - \vec{c} = \vec{0}$ …(ii)

By (i) and (ii), we conclude that $\vec{b} - \vec{c} = \vec{0}$ i.e., $\vec{b} = \vec{c}$.

Since two vectors can't be parallel and perpendicular simultaneously.

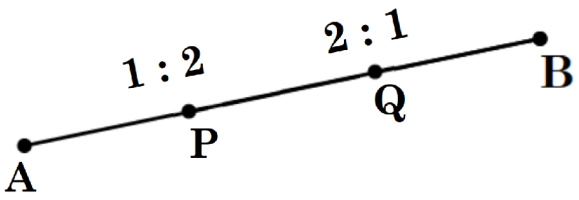
Remark In (b), $\vec{a} \neq 0$ is given, which should have been $\vec{a} \neq \vec{0}$.

25. A man needs to hang two lanterns on a straight wire whose end points have coordinates A(4, 1, –2) and B(6, 2, –3). Find the coordinates of the points where he hangs the lanterns such that these points trisect the wire AB.

Sol. Let P and Q be two points on the line (wire) joining A and B such that $AP = PQ = QB$.

That is, points P and Q trisect the line (wire) AB.

Clearly, $AP : PB = 1 : 2$ and $AQ : QB = 2 : 1$.



$$\therefore P\left(\frac{1\times 6+2\times 4}{1+2}, \frac{1\times 2+2\times 1}{1+2}, \frac{1\times(-3)+2\times(-2)}{1+2}\right)$$

$$\Rightarrow P\left(\frac{14}{3}, \frac{4}{3}, -\frac{7}{3}\right).$$

Similarly, $Q\left(\frac{2\times 6+1\times 4}{2+1}, \frac{2\times 2+1\times 1}{2+1}, \frac{2\times(-3)+1\times(-2)}{2+1}\right)$

$$\Rightarrow Q\left(\frac{16}{3}, \frac{5}{3}, -\frac{8}{3}\right).$$

SECTION C

There are 6 Short Answer questions in this section. Each is of 3 marks.

26. Find the value of 'a' for which $f(x) = \sqrt{3}\sin x - \cos x - 2a\,x + 6$ is decreasing in $\mathbb{R}$.

Sol. $f(x) = \sqrt{3}\sin x - \cos x - 2a\,x + 6$

$$\Rightarrow f'(x) = \sqrt{3}\cos x + \sin x - 2a$$

$$\Rightarrow f'(x) = 2\left(\frac{\sqrt{3}}{2}\cos x + \sin x \times \frac{1}{2}\right) - 2a$$

$$\Rightarrow f'(x) = 2\left(\sin\frac{\pi}{3}\cos x + \sin x \cos\frac{\pi}{3}\right) - 2a$$

$$\Rightarrow f'(x) = 2\sin\left(\frac{\pi}{3} + x\right) - 2a$$

As $f(x)$ is decreasing on $\mathbb{R}$ so, $f'(x) \le 0$ i.e., $2\sin\left(\frac{\pi}{3}+x\right) - 2a \le 0$

$$\therefore a \ge \sin\left(\frac{\pi}{3}+x\right) \quad \ldots(i)$$

We know that for all $x \in \mathbb{R}$, $\sin\left(\frac{\pi}{3}+x\right) \in [-1, 1]$ i.e., $-1 \le \sin\left(\frac{\pi}{3}+x\right) \le 1$ …(ii)

By (i) and (ii), we conclude that $a \ge 1$.

27. (a) Find $\int \frac{2x}{(x^2+3)(x^2-5)}\,dx$.

OR

(b) Evaluate $\int_1^4 \left(|x-2| + |x-4|\right)dx$.

Sol. (a) Put $x^2 = t \Rightarrow 2x\,dx = dt$.

Now $\int \frac{2x}{(x^2+3)(x^2-5)}\,dx = \int \frac{1}{(t+3)(t-5)}\,dt$

$$= \int \frac{1}{8}\left[\frac{1}{t-5} - \frac{1}{t+3}\right]dt$$

$$= \frac{1}{8}\left[\log|t-5| - \log|t+3|\right] + C$$

$$= \frac{1}{8}\left[\log|x^2-5| - \log|x^2+3|\right] + C = \frac{1}{8}\log\left|\frac{x^2-5}{x^2+3}\right| + C.$$

OR

(b) $\int_1^4 \left(|x-2|+|x-4|\right)dx = \int_1^4 |x-2|dx + \int_1^4 |x-4|dx = \int_1^2 |x-2|dx + \int_2^4 |x-2|dx + \int_1^4 -(x-4)dx$

$$= \int_1^2 -(x-2)dx + \int_2^4 (x-2)dx + \int_1^4 -(x-4)dx$$

$$= -\left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^4 - \left[\frac{(x-4)^2}{2}\right]_1^4$$

$$= -\left[0-\frac{1}{2}\right] + [2-0] - \left[0-\frac{9}{2}\right] = \frac{1}{2}+2+\frac{9}{2} = 5+2 = 7.$$

28. Find the particular solution of the differential equation $\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx + x\,dy = 0$ given that $y = \frac{\pi}{4}$, when $x = 1$.

Sol. Rewriting the D.E. $\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx + x\,dy = 0$, we get $\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

So, $v + x\frac{dv}{dx} = v - \sin^2 v \qquad \Rightarrow x\frac{dv}{dx} = -\sin^2 v$

$\Rightarrow \int -\text{cosec}^2 v\,dv = \int \frac{dx}{x}$

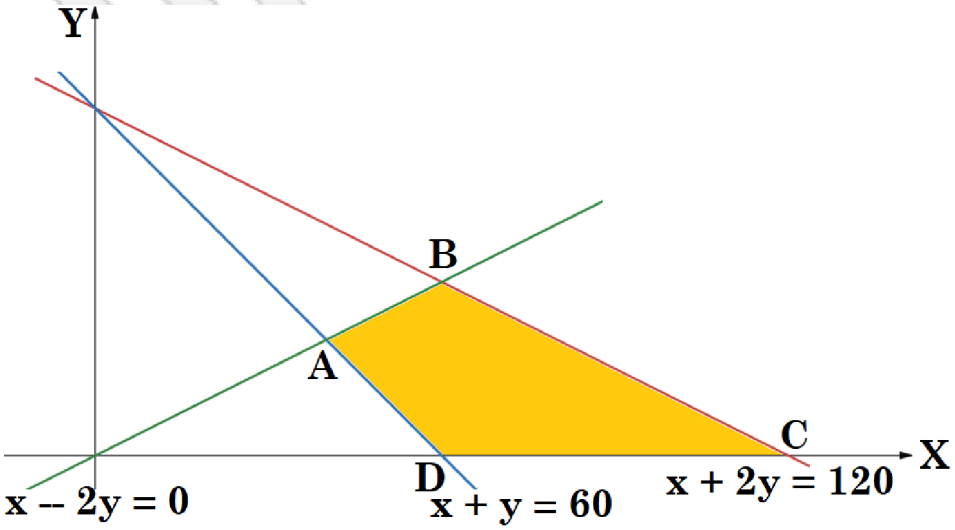
$\Rightarrow \cot v = \log|x| + C$ i.e., $\cot\frac{y}{x} = \log|x| + C$

It is given that $y = \frac{\pi}{4}$, when $x = 1$ so, $\cot\frac{\pi}{4} = \log|1| + C \Rightarrow C = 1 \qquad [\because \log|1| = 0$

Hence, the required particular solution is given by $\cot\left(\frac{y}{x}\right) = \log|x| + 1$.

29. In the Linear Programming Problem (LPP), find the point / points giving maximum value for $Z = 5x + 10y$ subject to constraints $x + 2y \le 120$, $x + y \ge 60$, $x - 2y \ge 0$; $x, y \ge 0$.

Sol. Consider the graph shown below.



The corner points are $A(40, 20)$, $B(60, 30)$, $C(120, 0)$ and $D(60, 0)$.

Corner Points	Value of Z
$A(40, 20)$	$5\times40+10\times20=400$
$B(60, 30)$	$5\times60+10\times30=600$ ← Maximum
$C(120, 0)$	$5\times120+10\times0=600$ ← Maximum
$D(60, 0)$	$5\times60+10\times0=300$

The maximum value of Z is '600' and it is obtained at $B(60, 30)$ and $C(120, 0)$.

Hence, $Z_{max}=600$ is obtained at all the points on the line-segment BC.

30. (a) If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ such that $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$, then find the angle between $\vec{a}$ and $\vec{b}$.

OR

(b) If $\vec{a}$ and $\vec{b}$ are unit vectors inclined with each other at an angle θ, then prove that $\frac{1}{2}|\vec{a}-\vec{b}|=\sin\frac{\theta}{2}$.

Sol. (a) $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ implies, $\vec{a}+\vec{b}=-\vec{c}$

Considering $(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=(-\vec{c}).(-\vec{c})$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}=|\vec{c}|^2$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta=|\vec{c}|^2$

$\Rightarrow 3^2+5^2+2\times3\times5\cos\theta=7^2$

$\Rightarrow \cos\theta=\frac{49-9-25}{2\times15}=\frac{1}{2} \quad \therefore \theta=\frac{\pi}{3}.$

OR

(b) Consider $|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b}).(\vec{a}-\vec{b})$

$\Rightarrow |\vec{a}-\vec{b}|^2=|\vec{a}|^2-2\vec{a}.\vec{b}+|\vec{b}|^2$

$\Rightarrow |\vec{a}-\vec{b}|^2=1^2-2|\vec{a}||\vec{b}|\cos\theta+1^2$ [$\because \vec{a}$ and $\vec{b}$ are unit vectors and θ is the angle betwen them

$\Rightarrow |\vec{a}-\vec{b}|^2=2(1-\cos\theta)=2\times2\sin^2\frac{\theta}{2}=4\sin^2\frac{\theta}{2}$

$\Rightarrow \sin^2\frac{\theta}{2}=\frac{1}{4}|\vec{a}-\vec{b}|^2$

Hence, $\frac{1}{2}|\vec{a}-\vec{b}|=\sin\frac{\theta}{2}$.

Alternatively, consider $\left[\frac{1}{2}|\vec{a}-\vec{b}|\right]^2=\frac{1}{4}(\vec{a}-\vec{b}).(\vec{a}-\vec{b})$

$\Rightarrow \left[\frac{1}{2}|\vec{a}-\vec{b}|\right]^2=\frac{|\vec{a}|^2+|\vec{b}|^2-2\vec{a}.\vec{b}}{4}$

$\Rightarrow \left[\frac{1}{2}|\vec{a}-\vec{b}|\right]^2=\frac{1^2+1^2-2|\vec{a}||\vec{b}|\cos\theta}{4}$

$$\Rightarrow \left[\frac{1}{2}\left|\vec{a}-\vec{b}\right|\right]^2 = \frac{2-2\times1\times1\cos\theta}{4} = \frac{2(1-\cos\theta)}{4}$$

$$\Rightarrow \left[\frac{1}{2}\left|\vec{a}-\vec{b}\right|\right]^2 = \frac{1}{2}\times 2\sin^2\frac{\theta}{2} = \sin^2\frac{\theta}{2}$$

Therefore, $\frac{1}{2}\left|\vec{a}-\vec{b}\right| = \sin\frac{\theta}{2}$.

31. (a) The probability that a student buys a colouring book is 0.7 and that she buys a box of colours is 0.2. The probability that she buys a colouring book, given that she buys a box of colours, is 0.3. Find the probability that the student

(i) buys both the colouring book and the box of colours.

(ii) buys a box of colours given that she buys the colouring book.

OR

(b) A person has a fruit box that contains 6 apples and 4 oranges. He picks out a fruit three times, one after the other, after replacing the previous one in the box. Find

(i) the probability distribution of the number of oranges he draws.

(ii) the expectation of the random variable (number of oranges).

Sol. (a) Let A and B denote the events that the student buys a colouring book and a box of colours respectively.

Given that $P(A) = 0.7$, $P(B) = 0.2$, $P(A \mid B) = 0.3$.

(i) $P(A \cap B) = P(A \mid B) \times P(B) = 0.3 \times 0.2 = 0.06$.

(ii) $P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{0.06}{0.7} = \frac{6}{70} = \frac{3}{35}$.

OR

(b) Let X denote the number of oranges drawn. So, X takes the values 0, 1, 2, 3.

So, $P(X=0) = \frac{3}{5}\times\frac{3}{5}\times\frac{3}{5} = \frac{27}{125}$, $P(X=1) = {}^3C_1\frac{2}{5}\times\frac{3}{5}\times\frac{3}{5} = \frac{54}{125}$, $P(X=2) = {}^3C_2\frac{2}{5}\times\frac{2}{5}\times\frac{3}{5} = \frac{36}{125}$,

$P(X=3) = \frac{2}{5}\times\frac{2}{5}\times\frac{2}{5} = \frac{8}{125}$

(i) The table of probability distribution is shown below.

X	0	1	2	3
P(X)	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

(ii) Expectation of the random variable $= \sum X\,P(X)$

$$= 0\times\frac{27}{125} + 1\times\frac{54}{125} + 2\times\frac{36}{125} + 3\times\frac{8}{125}$$

$$= \frac{54+72+24}{125} = \frac{150}{125} = \frac{6}{5}.$$

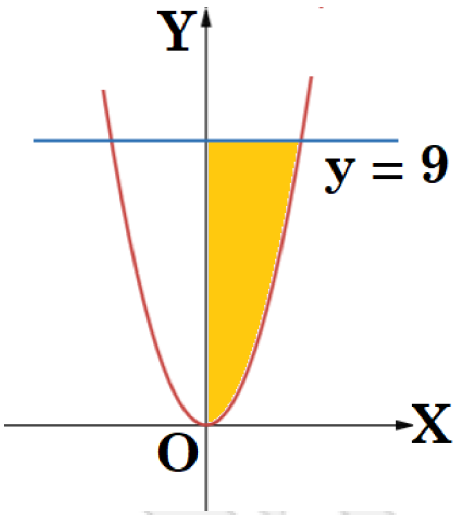
SECTION D

There are 4 Long Answer questions in this section. Each is of 5 marks.

32. Sketch a graph of $y = x^2$. Using integration, find the area of the region bounded by $y = 9$, $x = 0$ and $y = x^2$.

Sol. Consider the diagram shown for the sketch of $y = x^2$.

Note that, $y = x^2$ implies, $x = \pm\sqrt{y}$.



$$\text{Required area} = \int_0^9 \sqrt{y}\, dy$$

$$= \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^9$$

$$= \frac{2}{3}\left[9^{\frac{3}{2}} - 0\right]$$

$$= \frac{2}{3}\left[(9^{1/2})^3\right]$$

$$= \frac{2}{3}\left[3^3\right] = 18 \text{ Sq. units}.$$

33. A furniture workshop produces three types of furniture - chairs, tables and beds each day. On a particular day the total number of furniture pieces produced is 45. It was also found that production of beds exceeds that of chairs by 8, while the total production of beds and chairs together is twice the production of tables. Determine the units produced of each type of furniture, using matrix method.

Sol. Let x, y and z be the number of units produced of chairs, tables and beds respectively.

According to the question, $x + y + z = 45$, $z = x + 8$, $x + z = 2y$.

That is, $x + y + z = 45$, $-x + z = 8$, $x - 2y + z = 0$.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 45 \\ 8 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

The system of equations can be represented by $AX = B \Rightarrow X = A^{-1}B$ ...(i)

Now $|A| = 1(0+2) - 1(-1-1) + 1(2-0) = 2 + 2 + 2 = 6 \neq 0$ $\therefore A^{-1}$ exists.

Adjoint of matrix A, $\text{adj.}A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{bmatrix}$.

Inverse of matrix A, $A^{-1} = \frac{\text{adj.}A}{\det.(A)} = \frac{1}{6}\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{bmatrix}$

By (i), $X = \frac{1}{6}\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 45 \\ 8 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 66 \\ 90 \\ 114 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 19 \end{bmatrix}$

By equality of matrices, we get $x = 11$, $y = 15$, $z = 19$.

Hence, the number of units produced of chairs, tables and beds respectively are 11, 15 and 19.

34. (a) For a positive constant 'a', differentiate $a^{t+\frac{1}{t}}$ with respect to $\left(t+\frac{1}{t}\right)^a$, where t is a non-zero real number.

OR

(b) Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$, where a and b are constants.

Sol. (a) Let $y = a^{t+\frac{1}{t}}$ and $z = \left(t+\frac{1}{t}\right)^a$.

Now $\frac{dy}{dt} = a^{t+\frac{1}{t}} \times \log a \times \left(1-\frac{1}{t^2}\right)$ and $\frac{dz}{dt} = a\left(t+\frac{1}{t}\right)^{a-1} \times \left(1-\frac{1}{t^2}\right)$.

So, $\frac{dy}{dz} = \left(\frac{dy}{dt}\right) \div \left(\frac{dz}{dt}\right) = \frac{a^{t+\frac{1}{t}} \times \log a \times \left(1-\frac{1}{t^2}\right)}{a\left(t+\frac{1}{t}\right)^{a-1} \times \left(1-\frac{1}{t^2}\right)}$

$\therefore \frac{dy}{dz} = \frac{a^{t+\frac{1}{t}} \times \log a}{a\left(t+\frac{1}{t}\right)^{a-1}} = \frac{a^{t+\frac{1}{t}-1}}{\left(t+\frac{1}{t}\right)^{a-1}} \times \log a$.

Remark It has been mentioned that "t is a non-zero real number". Instead of this, the statement should be like this - "t is a non-zero real variable".

OR

(b) $y^x + x^y + x^x = a^b$

$\Rightarrow e^{\log x^y} + e^{\log y^x} + e^{\log x^x} = a^b$

$\Rightarrow e^{y\log x} + e^{x\log y} + e^{x\log x} = a^b$

Differentiating w.r.t. x both the sides, we get

$$e^{y\log x}\left(y\times\frac{1}{x}+\log x\times\frac{dy}{dx}\right)+e^{x\log y}\left(x\times\frac{1}{y}\times\frac{dy}{dx}+\log y.1\right)+e^{x\log x}\left(x\times\frac{1}{x}+\log x.1\right)=\frac{d}{dx}(a^b)$$

$$\Rightarrow x^y\left(\frac{y}{x}+\log x\times\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\times\frac{dy}{dx}+\log y\right)+x^x\left(1+\log x\right)=0$$

$$\Rightarrow\left(x^{y-1}y+x^y\log x\times\frac{dy}{dx}\right)+\left(y^{x-1}x\times\frac{dy}{dx}+y^x\log y\right)+x^x\left(1+\log x\right)=0$$

$$\Rightarrow\left(x^y\log x+y^{x-1}x\right)\times\frac{dy}{dx}=-\left[x^{y-1}y+y^x\log y+x^x\left(1+\log x\right)\right]$$

$$\therefore\frac{dy}{dx}=-\left[\frac{x^{y-1}y+y^x\log y+x^x\left(1+\log x\right)}{x^y\log x+y^{x-1}x}\right].$$

35. (a) Find the foot of the perpendicular drawn from the point (1, 1, 4) on the line $\frac{x+2}{5}=\frac{y+1}{2}=\frac{-z+4}{-3}$.

OR

(b) Find the point on the line $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-4}{3}$ at a distance of $2\sqrt{2}$ units from the point (–1, –1, 2).

Sol. (a) Consider $\frac{x+2}{5}=\frac{y+1}{2}=\frac{-z+4}{-3}=\lambda$.

The direction ratios of the line are 5, 2, 3.

Let A(1, 1, 4).

Assume that M be the foot of perpendicular on the line drawn from A.

Any arbitrary point on the line is $M(5\lambda-2,\ 2\lambda-1,\ 3\lambda+4)$.

Direction ratios of AM : $5\lambda-3, 2\lambda-2,\ 3\lambda$.

Since AM is perpendicular to the given line.

So by using $a_1a_2+b_1b_2+c_1c_2=0$ for $\perp$ lines, we get

$(5)(5\lambda-3)+(2)(2\lambda-2)+(3)(3\lambda)=0$

$\Rightarrow 25\lambda-15+4\lambda-4+9\lambda=0$

$\Rightarrow 38\lambda=19$

$\Rightarrow \lambda=\frac{1}{2}$.

$\therefore M\left(\frac{5}{2}-2,\ 2\times\frac{1}{2}-1,\ 3\times\frac{1}{2}+4\right)=M\left(\frac{1}{2},\ 0,\ \frac{11}{2}\right)$ is the required **foot of perpendicular**.

OR

(b) Let $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-4}{3}=\lambda$.

The coordinates of any random point on this line is $P(3\lambda+1,\ 2\lambda-1,\ 3\lambda+4)$.

Let Q(–1, –1, 2).

According to question, $PQ=\sqrt{(3\lambda+1+1)^2+(2\lambda-1+1)^2+(3\lambda+4-2)^2}=2\sqrt{2}$

On squaring both the sides, we get $(3\lambda+2)^2+(2\lambda)^2+(3\lambda+2)^2=8$

$\Rightarrow 2(9\lambda^2+4+12\lambda)+4\lambda^2=8$

$\Rightarrow 22\lambda^2+24\lambda+8=8$

$\Rightarrow 2\lambda(11\lambda+12)=0$

$\therefore \lambda=0,\ \lambda=-\frac{12}{11}$

Hence, the required points on the given line are $(3\times0+1,\ 2\times0-1,\ 3\times0+4)$ i.e., (1, –1, 4) and $\left(-\frac{36}{11}+1,\ -\frac{24}{11}-1,\ -\frac{36}{11}+4\right)$ i.e., $\left(-\frac{25}{11},\ -\frac{35}{11},\ \frac{8}{11}\right)$.

SECTION E

In this section there are 3 Case Study questions of 4 marks each.

36. A carpenter needs to make a wooden cuboidal box, closed from all sides, which has a square base and fixed volume. Since he is short of the paint required to paint the box on completion, he wants the surface area to be minimum.

On the basis of the above information, answer the following questions.

(i) Taking length = breadth = x m and height = y m, express the surface area (S) of the box in terms of x and its volume (V), which is constant.

(ii) Find $\frac{dS}{dx}$.

(iii) (a) Find a relation between x and y such that the surface area (S) is minimum.

OR

(iii) (b) If surface area (S) is constant, the volume $(V)=\frac{1}{4}(Sx-2x^3)$, x being the edge of base.

Show that volume (V) is maximum for $x=\sqrt{\frac{S}{6}}$.

Sol. **(i)** Volume of the box $(V)=x\times x\times y=x^2y \quad \Rightarrow y=\frac{V}{x^2}$

$\therefore$ Surface area $(S)=2(x\,x+x\,y+x\,y)=2x^2+4xy$

$\Rightarrow S=2x^2+\frac{4V}{x}$.

(ii) We have $S=2x^2+\frac{4V}{x} \quad \therefore \frac{dS}{dx}=4x-\frac{4V}{x^2}$.

(iii) (a) For $\frac{dS}{dx}=4x-\frac{4V}{x^2}=0$

$\Rightarrow \frac{4V}{x^2}=4x \quad \Rightarrow x^3=V \quad \therefore x=V^{\frac{1}{3}}$

Also, $\frac{d^2S}{dx^2}=4+\frac{8V}{x^3}$. Note that, $\left(\frac{d^2S}{dx^2}\right)_{\text{at } x=V^{1/3}}=4+\frac{8V}{V}=12>0$. So, S is minimum at $x=V^{\frac{1}{3}}$.

Now $y=\frac{V}{x^2} \quad \Rightarrow y=\frac{x^3}{x^2} \quad \therefore x=y$.

OR

(iii) (b) We have been given that, $V=\frac{1}{4}(Sx-2x^3)$, where S is constant

Now $\frac{dV}{dx}=\frac{1}{4}(S-6x^2)$ and $\frac{d^2V}{dx^2}=\frac{1}{4}(0-12x)=-3x$.

For $\frac{dV}{dx}=\frac{1}{4}(S-6x^2)=0 \quad \Rightarrow S=6x^2 \quad \therefore x=\sqrt{\frac{S}{6}}$

Note that $\left(\frac{d^2V}{dx^2}\right)_{\text{at } x=\sqrt{\frac{S}{6}}}=-3\left(\sqrt{\frac{S}{6}}\right)<0$.

Hence, V is maximum for $x=\sqrt{\frac{S}{6}}$.

37. Let A be the set of 30 students of class XII in a school. Let $f:A\to N$, N is a set of natural numbers such that function $f(x)=$ Roll Number of student x.

On the basis of the above information, answer the following questions.

(i) Is f a bijective function?

(ii) Give reasons to support your answer to **(i)**.

(iii) (a) Let R be a relation defined by the teacher to plan the seating arrangement of students in pairs, where $R=\{(x,y):x,\ y$ are Roll Numbers of students such that $y=3x\}$.

List all the elements of R. Is the relation R reflexive, symmetric and transitive? Justify your answer.

OR

(iii) (b) Let R be a relation defined by $R=\{(x,y):x,\ y$ are Roll Numbers of students such that $y=x^3\}$. List the elements of R. Is R a function? Justify your answer.

Sol. **(i)** Since f is not onto. Therefore, f is not bijective. Recall that, a bijection must be one-one and onto both.

(ii) Note that, the codomain of function f is N (set of natural numbers), whereas its range will be the set of natural numbers till 30, as we have only 30 students. That is, there will be many elements in the codomain (N) of f whose pre-images will not be there in the set A (domain). Hence, f is not onto.

However f is one-one, as every student will have a unique roll number allocated to them.

Recall that, a bijective function must be one-one and onto both.

Therefore, f is not bijective.

(iii) (a) $R = \{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15), (6, 18), (7, 21), (8, 24), (9, 27), (10, 30)\}$

Since $(a, a) \notin R \ \forall a \in A$. So, R is not reflexive.

Also, $(1, 3) \in R$ but $(3, 1) \notin R$. So, R is not symmetric.

Further $(1, 3) \in R$ and $(3, 9) \in R$ but $(1, 9) \notin R$. So, R is not transitive.

OR

(iii) (b) Given $R = \{(x, y) : x, y$ are Roll Numbers of students such that $y = x^3\}$.

So, $R = \{(1, 1), (2, 8), (3, 27)\}$.

To determine if R is a function from {1, 2, 3, …, 30} to {1, 2, 3, …, 30}, we need to check if R satisfies the definition of a function : for every x in domain, there is exactly one y in codomain such that $(x, y) \in R$.

Here the domain of R isn't explicitly stated, but since x and y are roll numbers of students, and R is defined over the roll numbers, it's **reasonable to assume the domain is the set of roll numbers that appear as first elements in R i.e., {1, 2, 3}, unless the statement given in the question intends the full set {1, 2, 3, …, 30} as the domain**.

In $R = \{(1, 1), (2, 8), (3, 27)\}$, the x values are 1, 2, 3 and each is paired with exactly one y value 1, 8, 27 respectively. There are no two ordered pairs in R having the same first element with different second elements. Each roll number x is uniquely mapped to its cube y, and all y values are within the possible range of roll numbers (1 to 30). **So, R is function on its listed domain {1, 2, 3}.**

However, if the intended domain is the full set {1, 2, 3, …, 30}, then R is not a function because for $x = 4, 5, 6, ..., 30$, there are no corresponding y values in R; since $y = x^3 > 30$ for $x \geq 4$.

Note that, when a relation R is defined by a rule and a set of ordered pairs is provided without specifying the domain, we often consider R as a function on the domain of the first element of the pairs.

38. A gardener wanted to plant vegetables in his garden. Hence he bought 10 seeds of brinjal plant, 12 seeds of cabbage plant and 8 seeds of radish plant. The shopkeeper assured him of germination probabilities of brinjal, cabbage and radish to be 25%, 35% and 40% respectively. But before he could plant the seeds, they got mixed up in the bag and he had to sow them randomly.



Radish



Cabbage



Brinjal

Based upon the above information, answer the following questions.

(i) Calculate the probability of a randomly chosen seed to germinate.

(ii) What is the probability that it is a cabbage seed, given that the chosen seed germinates?

Sol. Let E_1, E_2 and E_3 be the events of choosing brinjal, cabbage and radish respectively.

Let E be the event that the chosen seed germinates.

Now $P(E_1)=\frac{10}{30}$, $P(E_2)=\frac{12}{30}$, $P(E_3)=\frac{8}{30}$, $P(E\mid E_1)=25\%$, $P(E\mid E_2)=35\%$, $P(E\mid E_3)=40\%$.

(i) Using law of total probability, we get

$$P(E)=P(E_1)\times P(E\mid E_1)+P(E_2)\times P(E\mid E_2)+P(E_3)\times P(E\mid E_3)$$

$$\Rightarrow P(E)=\frac{10}{30}\times\frac{25}{100}+\frac{12}{30}\times\frac{35}{100}+\frac{8}{30}\times\frac{40}{100}$$

$$\Rightarrow P(E)=\frac{250+420+320}{30\times100}=\frac{990}{30\times100}=\frac{33}{100}=0.33=33\%.$$

(ii) By using Bayes' theorem, we get $P(E_2\mid E)=\frac{P(E_2)\times P(E\mid E_2)}{P(E)}$

$$\Rightarrow P(E_2\mid E)=\frac{\frac{12}{30}\times\frac{35}{100}}{\frac{990}{30\times100}}=\frac{12\times35}{990}=\frac{2\times7}{33}=\frac{14}{33}.$$

☑ Q.P. Code - 65/5/2

03. If A and B are two square matrices of the same order, then $(A+B)(A-B)$ is equal to

(a) $A^2-AB+BA-B^2$ (b) $A^2+AB-BA-B^2$

(c) $A^2-AB-BA-B^2$ (d) $A^2-B^2+AB+BA$

Sol. (a) $(A+B)(A-B)=AA-AB+BA-BB=A^2-AB+BA-B^2$.

08. If a matrix A is both symmetric and skew-symmetric, then A is a

(a) diagonal matrix (b) zero matrix

(c) non-singular matrix (d) scalar matrix

Sol. (b) A zero matrix (if it is square matrix) is both symmetric and skew-symmetric matrix.

11. $\int\frac{\cos 2x}{\sin^2 x\cos^2 x}dx$ is equal to

(a) $\cot x+\tan x+C$ (b) $-(\cot x+\tan x)+C$

(c) $-\cot x+\tan x+C$ (d) $\cot x-\tan x+C$

Sol. (b) $\int\frac{\cos 2x}{\sin^2 x\cos^2 x}dx=\int\frac{\cos^2 x-\sin^2 x}{\sin^2 x\cos^2 x}dx=\int\left[\operatorname{cosec}^2 x-\sec^2 x\right]dx=-\cot x-\tan x+C$.

12. If $\int_0^a\frac{1}{1+4x^2}dx=\frac{\pi}{8}$, then the value of 'a' is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) 4

Sol. (b) $\int_0^a\frac{1}{1+4x^2}dx=\frac{\pi}{8}\quad\Rightarrow\left[\frac{1}{2}\tan^{-1}2x\right]_0^a=\frac{\pi}{8}$

$$\Rightarrow\left[\tan^{-1}2a-\tan^{-1}0\right]=\frac{\pi}{4}$$

$$\Rightarrow 2a=\tan\frac{\pi}{4}=1\qquad\therefore a=\frac{1}{2}.$$

14. The integrating factor of the differential equation $\frac{dx}{dy}=\frac{x\log x}{\frac{2}{x}\log x-y}$ is

(a) $\frac{1}{8x}$ (b) e (c) $e^{\log x}$ (d) $\log x$

Sol. (d) Rewriting the D.E., $\frac{dy}{dx}=\frac{\frac{2}{x}\log x-y}{x\log x}=\frac{2}{x^2}-\frac{y}{x\log x}$ i.e., $\frac{dy}{dx}+\left(\frac{1}{x\log x}\right)y=\frac{2}{x^2}$.

Here $P(x)=\left(\frac{1}{x\log x}\right)$. Note that, $\int P(x)\,dx=\int\left(\frac{1}{x\log x}\right)dx=\log(\log x)$.

So, integrating factor $=e^{\int P(x)dx}=e^{\log(\log x)}=\log x$.

21. Find the domain of $\sec^{-1}(2x+1)$.

Sol. $\sec^{-1}(2x+1)$ is defined when $(2x+1)\le -1\cup(2x+1)\ge 1$

$\Rightarrow x\le -1\cup x\ge 0$

Therefore, the domain is $x\in(-\infty,-1]\cup[0,\infty)$.

22. The radius of a cylinder is decreasing at a rate of 2 cm/s and the altitude is increasing at the rate of 3 cm/s. Find the rate of change of volume of this cylinder when its radius is 4 cm and altitude is 6 cm.

Sol. Given that $\frac{dr}{dt}=-2$ cm/s and $\frac{dh}{dt}=3$ cm/s.

As the volume of cylinder is, $V=\pi r^2h$

$\Rightarrow\frac{dV}{dt}=\pi\left[r^2\frac{dh}{dt}+2rh\frac{dr}{dt}\right]$

When $r=4$ cm, $h=6$ cm, $\frac{dV}{dt}=\pi\left[4^2\times3+2\times4\times6\times(-2)\right]=\pi[48-96]=-48\pi$ cm^3/s.

Hence, the volume is decreasing at the rate of 48π cm^3/s.

26. Show that $f(x)=\tan^{-1}(\sin x+\cos x)$ is an increasing function in $\left[0,\frac{\pi}{4}\right]$.

Sol. $f(x)=\tan^{-1}(\sin x+\cos x)$, $x\in\left[0,\frac{\pi}{4}\right]$

$\therefore f'(x)=\frac{1}{1+(\sin x+\cos x)^2}\times(\cos x-\sin x)=\frac{\cos x-\sin x}{1+\sin^2x+\cos^2x+2\sin x\cos x}$

$\Rightarrow f'(x)=\frac{\cos x-\sin x}{2+\sin 2x}$

As $\cos x\ge\sin x$ i.e., $\cos x-\sin x\ge 0$ in $x\in\left[0,\frac{\pi}{4}\right]$ so, $f'(x)=\frac{\cos x-\sin x}{2+\sin 2x}\ge 0$ in $x\in\left[0,\frac{\pi}{4}\right]$.

Note that, $2+\sin 2x$ is always positive as $-1\le\sin 2x\le 1\ \forall\ x\in\mathbb{R}$.

So, f is increasing in $x\in\left[0,\frac{\pi}{4}\right]$.

28. Find the particular solution of the differential equation $\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\left(\frac{y}{x}\right)=0$, given that $y=0$ when $x=1$.

Sol. Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

So, the D.E. $\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$ becomes $v + x\frac{dv}{dx} - v + \text{cosec}\, v = 0$

$\Rightarrow x\frac{dv}{dx} = -\text{cosec}\, v$

$\Rightarrow \int -\sin v\, dv = \int \frac{dx}{x}$

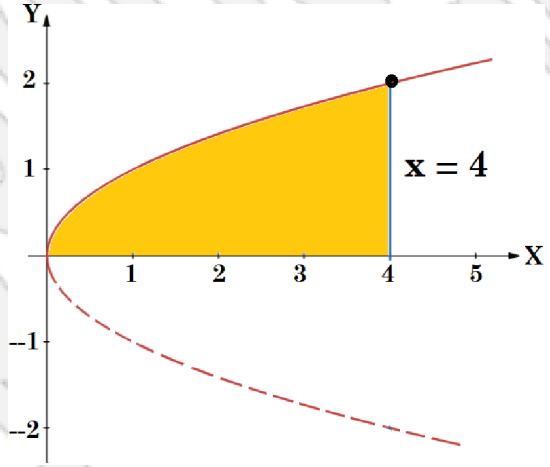
$\Rightarrow \cos v = \log|x| + C$ i.e., $\cos\frac{y}{x} = \log|x| + C$.

As $y = 0$ when $x = 1$ so, $\cos 0 = \log|1| + C \quad \Rightarrow C = 1 \quad [\because \log|1| = 0$

Hence, the required particular solution is $\cos\left(\frac{y}{x}\right) = \log|x| + 1$.

32. Draw a rough sketch of the curve $y = \sqrt{x}$. Using integration, find the area of the region bounded by the curve $y = \sqrt{x}$, $x = 4$ and x-axis, in the first quadrant.

Sol. For $y = \sqrt{x}$, we can get $y^2 = x$ as reference.
Refer the diagram shown.



Required area $= \int_0^4 \sqrt{x}\, dx$

$= \frac{2}{3}\left[x^{3/2}\right]_0^4$

$= \frac{2}{3}\left[4^{3/2} - 0\right]$

$= \frac{16}{3}$ Sq. units.

33. An amount of ₹10000 is put into three investments at the rate of 10%, 12% and 15% per annum. The combined annual income of all three investments is ₹1310, however the combined annual income of the first and the second installments is ₹190 short of the income from the third. Use matrix method and find the investment amount in each at the beginning of the year.

Sol. Let the amount invested be x, y and z (in ₹) at the respective rate of 10%, 12% and 15%.
So, $x + y + z = 10000$, $x \times 10\% + y \times 12\% + z \times 15\% = 1310$, $x \times 10\% + y \times 12\% = z \times 15\% - 190$.
That is, $x + y + z = 10000$, $10x + 12y + 15z = 131000$, $10x + 12y - 15z = -19000$.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 10 & 12 & 15 \\ 10 & 12 & -15 \end{bmatrix}$, $B = \begin{bmatrix} 10000 \\ 131000 \\ -19000 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Clearly, $AX = B \Rightarrow X = A^{-1}B$ ...(i)

Here $|A| = 1(-180 - 180) - 1(-150 - 150) + 1(120 - 120) = -360 + 300 + 0 = -60 \neq 0 \ \therefore A^{-1}$ exists.

Adjoint of matrix A, $\text{adj.}A = \begin{bmatrix} -360 & 27 & 3 \\ 300 & -25 & -5 \\ 0 & -2 & 2 \end{bmatrix}$

Inverse of matrix A, $A^{-1} = \frac{\text{adj. A}}{|A|} = \frac{1}{60}\begin{bmatrix} 360 & -27 & -3 \\ -300 & 25 & 5 \\ 0 & 2 & -2 \end{bmatrix}$

By (i), we get $X = \frac{1}{60}\begin{bmatrix} 360 & -27 & -3 \\ -300 & 25 & 5 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 10000 \\ 131000 \\ -19000 \end{bmatrix} = \frac{1}{60}\times 1000\begin{bmatrix} 360 & -27 & -3 \\ -300 & 25 & 5 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 10 \\ 131 \\ -19 \end{bmatrix}$

$\Rightarrow X = \frac{100}{6}\begin{bmatrix} 120 \\ 180 \\ 300 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 3000 \\ 5000 \end{bmatrix}$

By equality of matrices, we get $x = 2000,\ y = 3000,\ z = 5000$.

Hence, the investment amounts are ₹2000 at 10% per annum, ₹3000 at 12% per annum, ₹5000 at 15% per annum.

☑ Q.P. Code - 65/5/3

03. If $A = kB$, where A and B are two square matrices of order n and k is scalar, then

(a) $|A| = k|B|$ (b) $|A| = k^n|B|$ (c) $|A| = k + |B|$ (d) $|A| = |B|^k$

Sol. (b) $|A| = |kB|$ gives, $|A| = k^n|B|$, where n is order of matrix B.

08. Let A and B be two matrices of suitable orders. Then, which of the following is **not** correct?

(a) $(A')' = A$ (b) $(kA)' = kA'$, k is a scalar

(c) $(A' + B')' = A + B$ (d) $(AB)' = A'B'$

Sol. (d) Note that, $(AB)' = B'A'$.

Therefore, $(AB)' = A'B'$ is **not** correct.

11. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$ is equal to

(a) $2(\sin x + x\cos\theta) + C$ (b) $2(\sin x - x\cos\theta) + C$

(c) $2(\sin x + \sin\theta) + C$ (d) $2(\sin x - x\sin\theta) + C$

Sol. (a) $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2\theta - 1)}{\cos x - \cos\theta} dx$

$= \int \frac{2(\cos^2 x - \cos^2\theta)}{\cos x - \cos\theta} dx$

$= 2\int (\cos x + \cos\theta)\, dx$

$= 2(\sin x + x\cos\theta) + C.$

12. $\int_0^1 \frac{2x}{5x^2 + 1} dx$ is equal to

(a) $\frac{1}{5}\log 6$ (b) $\frac{1}{5}\log 5$ (c) $\frac{1}{2}\log 6$ (d) $\frac{1}{2}\log 5$

Sol. (a) $\int_0^1 \frac{2x}{5x^2 + 1} dx = \int_0^1 \frac{2x}{5\left(x^2 + \frac{1}{5}\right)} dx = \frac{1}{5}\left[\log\left|x^2 + \frac{1}{5}\right|\right]_0^1$

$$= \frac{1}{5}\left[\log\frac{6}{5} - \log\frac{1}{5}\right] = \frac{1}{5}\left[\log\left(\frac{\frac{6}{5}}{\frac{1}{5}}\right)\right] = \frac{1}{5}\log 6 .$$

14. The integrating factor of the differential equation $\frac{dx}{dy} = \frac{-(1+\sin x)}{x + y\cos x}$ is

(a) $\log\cos x$ (b) $1+\sin x$ (c) $e^{(1+\sin x)}$ (d) $e^{\log\cos x}$

Sol. (b) Rewriting the D.E., we get $\frac{dy}{dx} = -\frac{x + y\cos x}{1+\sin x}$ i.e., $\frac{dy}{dx} + \left(\frac{\cos x}{1+\sin x}\right)y = -\frac{x}{1+\sin x}$

Here $P(x) = \left(\frac{\cos x}{1+\sin x}\right)$. Note that, $\int P(x)dx = \int\left(\frac{\cos x}{1+\sin x}\right)dx = \log(1+\sin x)$.

Therefore, integrating factor is $= e^{\int P(x)dx} = e^{\log(1+\sin x)} = (1+\sin x)$.

21. Find the domain of $\sin^{-1}(x^2 - 3)$.

Sol. For the given function to be defined, we must have $-1 \le (x^2 - 3) \le 1$ i.e., $2 \le x^2 \le 4$

That implies, $-2 \le x \le -\sqrt{2} \cup \sqrt{2} \le x \le 2$.

So, the domain of $\sin^{-1}(x^2 - 3)$ is $x \in \left[-2, -\sqrt{2}\right] \cup \left[\sqrt{2}, 2\right]$.

22. Let the volume of a metallic hollow sphere be constant. If the inner radius increases at the rate of 2 cm/s, then find the rate of increase of the outer radius when the radii are 2 cm and 4 cm respectively.

Sol. Let V denote the constant volume of hollow sphere with outer radius R and inner radius r.

Then, $V = \frac{4\pi}{3}(R^3 - r^3)$

$\Rightarrow \frac{dV}{dt} = \frac{4\pi}{3}\left[3R^2\frac{dR}{dt} - 3r^2\frac{dr}{dt}\right]$

$\Rightarrow 0 = \frac{4\pi}{3}\times 3\left[R^2\frac{dR}{dt} - r^2\times 2\right]$ $\left[\because \frac{dr}{dt} = 2 \text{ cm/s}\right.$

When $r = 2$ cm, $R = 4$ cm, $\frac{0}{4\pi} = \left[4^2\frac{dR}{dt} - 2^2\times 2\right]$

$\Rightarrow 16\times\frac{dR}{dt} - 8 = 0$ i.e., $\frac{dR}{dt} = \frac{8}{16} = 0.5$ cm/s.

Therefore, the rate of increase of the outer radius is 0.5 cm/s.

26. Find the interval (s) in which the function $f(x) = \sin 3x - \cos 3x$, $0 < x < \frac{\pi}{2}$ is strictly increasing.

Sol. For $f(x) = \sin 3x - \cos 3x$, $0 < x < \frac{\pi}{2}$

$\Rightarrow f'(x) = 3\cos 3x + 3\sin 3x = 3(\cos 3x + \sin 3x)$

For critical points, $f'(x) = 0$ i.e., $3(\cos 3x + \sin 3x) = 0$

$\Rightarrow \tan 3x = -1$

$\Rightarrow 3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$

$\Rightarrow x = \frac{1}{3}\times\frac{3\pi}{4}, \frac{1}{3}\times\frac{7\pi}{4}, \frac{1}{3}\times\frac{11\pi}{4}, \ldots$

$\therefore x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$.

Note that, $f'(x) = 3(\cos 3x + \sin 3x) > 0$ in $x \in \left(0, \frac{\pi}{4}\right)$.

Hence, f is strictly increasing in $x \in \left(0, \frac{\pi}{4}\right)$.

28. Solve the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$.

Sol. Rewriting the given D.E., we get $\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = \frac{y}{x}\cos\left(\frac{y}{x}\right) + 1$ i.e., $\frac{dy}{dx} = \frac{y}{x} + \frac{1}{\cos\left(\frac{y}{x}\right)}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

The D.E. becomes $v + x\frac{dv}{dx} = v + \frac{1}{\cos v}$ i.e., $x\frac{dv}{dx} = \frac{1}{\cos v}$

That is, $\int \cos v\, dv = \int \frac{dx}{x}$

$\Rightarrow \sin v = \log|x| + C$

Hence, the required solution is $\sin\left(\frac{y}{x}\right) = \log|x| + C$.

32. In a rough sketch, mark the region bounded by $y = 1 + |x+1|$, $x = -2$, $x = 2$ and $y = 0$. Using integration, find the area of the marked region.

Sol. We have $y = 1 + |x+1| = \begin{cases} 2 + x, & \text{if } x \geq -1 \\ -x, & \text{if } x < -1 \end{cases}$ ...(i) and, $x = 2$, $x = -2$ and $y = 0$ (i.e., x-axis).

$\therefore$ Required area $= \int_{-2}^{2} y\, dx$

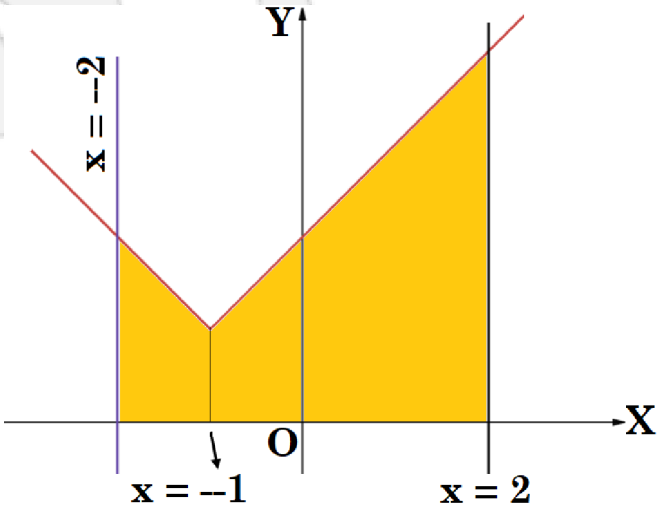
$\Rightarrow \quad = \int_{-2}^{-1} y dx + \int_{-1}^{2} y dx$

$\Rightarrow \quad = \int_{-2}^{-1} -x dx + \int_{-1}^{2} (2+x) dx$

$\Rightarrow \quad = -\frac{1}{2}\left[x^2\right]_{-2}^{-1} + \frac{1}{2}\left[(2+x)^2\right]_{-1}^{2}$

$\Rightarrow \quad = -\frac{1}{2}[1-4] + \frac{1}{2}[16-1]$

$\Rightarrow \quad = \frac{3}{2} + \frac{15}{2} = 9$ Sq. units.



33. Three students run on a racing track such that their speeds add up to 6 km/h. However, double the speed of the third runner added to the speed of the first results in 7 km/h. If thrice the speed of the first runner is added to the original speeds of the other two, the result is 12 km/h. Using matrix method, find the original speed of each runner.

Sol. Let x be the speed of the first runner, y be the speed of the second runner, z be the speed of the third runner; keeping in mind that all the speeds are in km/h.

According to question, $x + y + z = 6$, $x + 2z = 7$, $3x + y + z = 12$.

Let $A=\begin{bmatrix}1&1&1\\1&0&2\\3&1&1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}6\\7\\12\end{bmatrix}$.

Note that, $AX=B \Rightarrow X=A^{-1}B$ …(i)

Now $|A|=1(0-2)-1(1-6)+1(1-0)=-2+5+1=4\neq 0 \therefore A^{-1}$ exists.

Adjoint of matrix A, $\text{adj.}A=\begin{bmatrix}-2&0&2\\5&-2&-1\\1&2&-1\end{bmatrix}$

Inverse of matrix A, $A^{-1}=\frac{\text{adj.}A}{|A|}=\frac{1}{4}\begin{bmatrix}-2&0&2\\5&-2&-1\\1&2&-1\end{bmatrix}$

By (i), $X=\frac{1}{4}\begin{bmatrix}-2&0&2\\5&-2&-1\\1&2&-1\end{bmatrix}\begin{bmatrix}6\\7\\12\end{bmatrix} \Rightarrow X=\frac{1}{4}\begin{bmatrix}-12+0+24\\30-14-12\\6+14-12\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\1\\2\end{bmatrix}$

By equality of the matrices, we get $x=3, y=1, z=2$.

Hence, the speed of first runner : 3 km/h, the speed of second runner : 1 km/h and the speed of third runner : 2 km/h.

With regards,
O.P. GUPTA
Author & Math Mentor
Indira Award Winner
The O.P. Gupta Advanced Math Classes
@ **Think Academy**, Najafgarh
Telegram / WhatsApp : +919650350480
YouTube.com/@theopgupta

The Series ZXW5Y ♦ Q.P. Code 65/5/1; 65/5/2; 65/5/3 is of Punjab, Maharashtra, Kerala, Andhra Pradesh, Telangana, Foreign region.

© All Rights Reserved with O.P. GUPTA

Disclaimer : All care has been taken while preparing this solution draft. Solutions have been verified by prominent academicians having vast knowledge and experience in teaching of Math. Still if any error is found, please bring it to our notice.

Kindly forward your concerns/feedbacks through **message** on **WhatsApp @ +919650350480** or mail at **iMathematicia@gmail.com**